LECTURE 35 SIGMA NOTATION AND LIMITS OF FINITE SUMS

Motivated by approximating the area under the graph of a function using finite sums, today we study a more general presentation of sums involving large number of pieces. First, consider the example in the quiz, i.e. estimate the area under $f(x)$ with 4 subintervals on $[0,\pi]$ using the left endpoint rule. We end up with

$$Area = \Delta x \left(f(0) + f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right) \right).$$

Sigma Notation and Indexing

While four terms here are manageable, one wishes to improve the accuracy of the approximation by using more subintervals. Then, we just have bigger and bigger number of terms in a sum, a taxing task to write with your hands, even when you program. We need a simpler yet effective way of describing "adding a lot but still finite number of things up". This yields the sigma notation,

$$\sum_{k=1}^{n} a_k.$$

Here, a_k is called the **summand** (like integrand), which is the expression of the k^{th} term in the long list of terms. k is the index of terms we are adding.1 is the **lower limit** of the sum, the **index** of the **first** term you add. n is the **upper limit** of the sum, the **index** of the **last** term you add. Suppose we have a list of numbers $\{a_1, a_2, a_3, a_4, a_5\}$. Then,

$$\sum_{k=1}^{3} a_k = a_1 + a_2 + a_3$$

$$\sum_{k=2}^{4} a_k = a_2 + a_3 + a_4$$

Let's see some specific examples of a_k. Note that the index k does not have to start at $k = 1$ necessarily.

Example. (Classic examples)

(1)

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5.$$

(2)

$$\sum_{k=1}^{3} (-1)^k k = (-1)^1 1 + (-1)^2 2 + (-1)^3 3.$$

(3)

$$\sum_{k=4}^{5} \frac{k^2}{k-1} = \frac{4^2}{4-1} + \frac{5^2}{5-1}.$$

Remark. Carl Friedrich Gauss managed to add up

$$(1 + 2 + \cdots + 100) = \sum_{k=1}^{100} k$$

by observing that $a_k + a_{101-k} = 101$ holds for every $k = 1, \ldots, 50$. Therefore, the sum is $50 \times 101 = 5050$. There are many versions of this story, but one thing stays true – he was seven when he noticed this.

Notice that we are given the pattern of a_k here for each sum. Writing out what the sum actually is is quite simple from this end. The harder part is to recognize the pattern from a sequence of numbers and then express it as a sum, i.e. find the general expression for a_k.

Example. Write $1+3+5+7+9$ in many ways, i.e. the lower limit does not have to start at $k=1$.

$$\text{(Starting with } k=0\text{)} \quad \sum_{k=0}^{4}(2k+1)$$

$$\text{(Starting with } k=1\text{)} \quad \sum_{k=1}^{5}(2k-1)$$

$$\text{(Starting with } k=2\text{)} \quad \sum_{k=2}^{6}(2k-3)$$

$$\text{(Starting with } k=-3\text{)} \quad \sum_{k=-3}^{1}(2k+7)$$

Linearity of Summation

Sums are also a linear process, that is

$$\sum_{k=1}^{n}(ca_k+db_k)=c\sum_{k=1}^{n}a_k+d\sum_{k=1}^{n}b_k$$

where c,d are constant (independent of k). In particular, we collect the following three rules,

(1)

$$\sum_{k=1}^{n}(a_k \pm b_k)=\sum_{k=1}^{n}a_k \pm \sum_{k=1}^{n}b_k$$

(2)

$$\sum_{k=1}^{n}ca_k=c\sum_{k=1}^{n}a_k$$

(3)

$$\sum_{k=1}^{n}c=c\sum_{k=1}^{n}1=c\cdot\text{(adding up the number 1, } n \text{ times)}=cn.$$

Example. When you see a complicated-looking summand, yet its terms are all added or subtracted, you can divide and conquer.

$$\sum_{k=1}^{n}\left(3k-k^2\right)=3\sum_{k=1}^{n}k-\sum_{k-1}^{n}k^2.$$

You can factor out negative numbers, too,

$$\sum_{k=1}^{n}(-a_k)=\sum_{k=1}^{n}(-1)\,a_k=(-1)\sum_{k=1}^{n}a_k=-\sum_{k=1}^{n}a_k.$$

Special Sums

Lastly, we look at three specific sums whose formula can be derived.

$$\sum_{k=1}^{n}k=\frac{n\,(n+1)}{2}.$$

This formula is inspired by Gauss's way of adding consecutive integers. Consider adding up twice and then divide by 2, but in a particular order,

$$1+2+\cdots+n$$

$$n+(n-1)+\cdots+1$$

Note that if we add up every k^{th} term in each sequence, we always end up with $(n+1)$, and we have n of them. Thus, the sum of one sequence is $\frac{n(n+1)}{2}$. This approach avoids the argument of considering n odd or even.

We also have sum of squares and cubes of consecutive integers,

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$
$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

Note that the sum of cubes is exactly the square of sum of the first n integers. This is a remarkable fact from arithmetic.

Relation to Integration

Our original goal is to make finer and finer partitions to improve the accuracy of our approximations of the area under the graph of a function. This process involves having more and more terms in the sum, whose summand will follow a pattern. Eventually, we want to take the number of subintervals to infinity, and see if the sum yields a number.

Example. Find the limiting value of **lower sum** approximations to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity.

Solution. Suppose we chop down $[0, 1]$ using n points. Then, each subinterval has length $\Delta x = \frac{1-0}{n} = \frac{1}{n}$. Our partition is

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}.$$

Since the function is decreasing on $[0, 1]$, a **lower sum** estimate would be using the right endpoint (to underestimate, or to have all rectangles be inscribed in the region R). We find

$$\begin{aligned}
\mathit{Area}(n) &= \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right) + f\left(\frac{n}{n}\right)\right) \\
&= \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) \\
&= \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \\
&= \frac{1}{n}\sum_{k=1}^{n} \left(1 - \left(\frac{k}{n}\right)^2\right) \\
&= \frac{1}{n}\sum_{k=1}^{n} \left(1 - \frac{k^2}{n^2}\right) \\
&= \frac{1}{n}\left[\left(\sum_{k=1}^{n} 1\right) - \left(\frac{1}{n^2}\sum_{k=1}^{n} k^2\right)\right] \\
&= \frac{1}{n}\left[n - \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right] \\
&= 1 - \frac{2n^2 + 3n + 1}{6n^2}
\end{aligned}$$

that is, this is the area of region R given that we use n subintervals. Give me an n and I can tell you what the approximate area is.

Now, the goal is to take a limit $n \to \infty$. We find

$$\lim_{n\to\infty}\left(1 - \frac{2n^2 + 3n + 1}{6n^2}\right) = 1 - \frac{1}{3} = \frac{2}{3}$$

as the second term requires one more step of dividing top and bottom by n^2.

In fact, you can try getting the **upper sum** approximation, here, using the left endpoint rule. You will find that the limit is also $\frac{2}{3}$. In fact, you can show that any finite approximation will give this number, while being sandwiched between the upper and lower sum approximations. Hence, we **define** the area of R as this limiting value.